

$(3, 1)^*$ -choosability of planar graphs without adjacent short cycles

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Abstract

A list assignment of a graph G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $(L, d)^*$ -coloring is a mapping π that assigns a color $\pi(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most d neighbors of v receive color $\pi(v)$. A graph G is said to be $(k, d)^*$ -choosable if it admits an $(L, d)^*$ -coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. In 2001, Lih et al. [6] proved that planar graphs without 4- and l -cycles are $(3, 1)^*$ -choosable, where $l \in \{5, 6, 7\}$. Later, Dong and Xu [3] proved that planar graphs without 4- and l -cycles are $(3, 1)^*$ -choosable, where $l \in \{8, 9\}$.

There exist planar graphs containing 4-cycles that are not $(3, 1)^*$ -choosable (Crown, Crown and Woodall, 1986 [1]). This partly explains the fact that in all above known sufficient conditions for the $(3, 1)^*$ -choosability of planar graphs the 4-cycles are completely forbidden. In this paper we allow 4-cycles nonadjacent to relatively short cycles. More precisely, we prove that every planar graph without 4-cycles adjacent to 3- and 4-cycles is $(3, 1)^*$ -choosable. This is a common strengthening of all above mentioned results. Moreover as a consequence we give a partial answer to a question of Xu and Zhang [11] and show that every planar graph without 4-cycles is $(3, 1)^*$ -choosable.

Keyword: Planar graphs; Improper choosability; Cycle.

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. For a graph G , we use $V(G)$, $E(G)$, $|G|$, $|E(G)|$ and $\delta(G)$ to denote its vertex set, edge set, order, size and minimum degree, respectively. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G . If there is no confusion about the context, we write $N(v)$ for $N_G(v)$.

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A k -coloring of G is a mapping π from $V(G)$ to a color set $\{1, 2, \dots, k\}$ such that $\pi(x) \neq \pi(y)$ for any adjacent vertices x and y . A graph is k -colorable if it has a k -coloring. Cowen, Cowen, and Woodall [1] considered *defective* colorings of graphs. A graph G is said to be d -improper k -colorable, or simply, $(k, d)^*$ -colorable, if the vertices of G can be colored with k colors in such a way that each vertex has at most d neighbors receiving the same color as itself. Obviously, a $(k, 0)^*$ -coloring is an ordinary proper k -coloring.

A *list assignment* of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An L -coloring with impropriety of integer d , or simply an $(L, d)^*$ -coloring, of G is a mapping π that assigns a color $\pi(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most d neighbors of v receive color $\pi(v)$. A graph is k -choosable with impropriety of integer d , or simply $(k, d)^*$ -choosable, if there exists an $(L, d)^*$ -coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. Clearly, a $(k, 0)^*$ -choosable is the ordinary k -choosability introduced by Erdős, Rubin and Taylor [5] and independently by Vizing [10].

The concept of list improper coloring was independently introduced by Škrekovski [7] and Eaton and Hull [4]. They proved that every planar graph is $(3, 2)^*$ -choosable and every outerplanar graph is $(2, 2)^*$ -choosable. These are both improvement of the results showed in [1] which say that every planar graph is $(3, 2)^*$ -colorable and every outerplanar graph is $(2, 2)^*$ -colorable. Let $g(G)$ denote the *girth* of a graph G , i.e., the length of a shortest cycle in G . The $(k, d)^*$ -choosability of planar graph G with given $g(G)$ has been studied by Škrekovski in [9]. He proved that every planar graph G is $(2, 1)^*$ -choosable if $g(G) \geq 9$, $(2, 2)^*$ -choosable if $g(G) \geq 7$, $(2, 3)^*$ -choosable if $g(G) \geq 6$, and $(2, d)^*$ -choosable if $d \geq 4$ and $g(G) \geq 5$. Recently, Cushing and Kierstead [2] proved that every planar graph is $(4, 1)^*$ -choosable. So it would be interesting to investigate the sufficient conditions of $(3, 1)^*$ -choosability of subfamilies of planar graphs where some families of cycles are forbidden. Škrekovski proved in [8] that every planar graph without 3-cycles is $(3, 1)^*$ -choosable. Lih et al. [6] proved that planar graphs without 4- and l -cycles are $(3, 1)^*$ -choosable, where $l \in \{5, 6, 7\}$. Later, Dong and Xu [3] proved that planar graphs without 4- and l -cycles are $(3, 1)^*$ -choosable, where $l \in \{8, 9\}$. Moreover, Xu and Zhang [11] asked the following question:

Question 1 *Is it true that every planar graph without adjacent triangles is $(3, 1)^*$ -choosable?*

Recall that there is a planar graph containing 4-cycles that is not $(3, 1)^*$ -colorable [1]. Therefore, while describing $(3, 1)^*$ -choosability planar graphs, one must impose these or those restrictions on 4-cycles. Note that in all previously known sufficient conditions for the $(3, 1)^*$ -choosability of planar

graphs, the 4-cycles are completely forbidden. In this paper we allow 4-cycles, but disallow them to have a common edge with relatively short cycles.

The purpose of this paper is to prove the following

Theorem 1 *Every planar graph without 4-cycles adjacent to 3- and 4-cycles is $(3, 1)^*$ -choosable.*

Clearly, Theorem 1 implies Corollary 1 which is a common strengthening of the results in [6, 3].

Corollary 1 *Every planar graph without 4-cycles is $(3, 1)^*$ -choosable.*

Moreover, Theorem 1 partially answers Question 1, since adjacent triangles can be regarded as a 4-cycle adjacent to a 3-cycle.

2 Notation

A vertex of degree k (resp. at least k , at most k) will be called a k -vertex (resp. k^+ -vertex, k^- -vertex). A similar notation will be used for cycles and faces. A *triangle* is synonymous with a 3-cycle. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \cdots u_n]$ if u_1, u_2, \dots, u_n are the boundary vertices of f in cyclic order. For any $v \in V(G)$, we let $v_1, v_2, \dots, v_{d(v)}$ denote the neighbors of v in a cyclic order. Let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \dots, d(v)$, where indices are taken modulo $d(v)$. Moreover, we let $t(v)$ denote the number of 3-faces incident to v and let $n_3(v)$ denote the number of 3-vertices adjacent to v .

An m -face $f = [v_1 v_2 \cdots v_m]$ is called an (a_1, a_2, \dots, a_m) -face if the degree of the vertex v_i is a_i for $i = 1, 2, \dots, m$. Suppose v is a 4-vertex incident to a 4⁻-face f and adjacent to two 3-vertices not on $b(f)$. If $d(f) = 3$, then we call v a *light* 4-vertex. Otherwise, we call v a *soft* 4-vertex if $d(f) = 4$. A vertex v is called an \mathcal{S} -vertex if it is either a 3-vertex or a light 4-vertex. Moreover, we say a 3-face $f = [v_1 v_2 v_3]$ is an $(a_1, *, a_3)$ -face if $d(v_i) = a_i$ for each $i \in \{1, 3\}$ and v_2 is an \mathcal{S} -vertex. Suppose v is a 5-vertex incident to two 3-faces $f_1 = [vv_1 v_2]$ and $f_3 = [vv_3 v_4]$. Let v_5 be the neighbour of v not belonging to the 3-faces. If $d(v_5) = 3$ and f_1 is a $(5, *, 4)$ -face, then we call v a *bad* 5-vertex.

For all figures in the following section, a vertex is represented by a solid circle when all of its incident edges are drawn; otherwise it is represented by a hollow circle. Moreover, we use a hollow square to denote an \mathcal{S} -vertex.

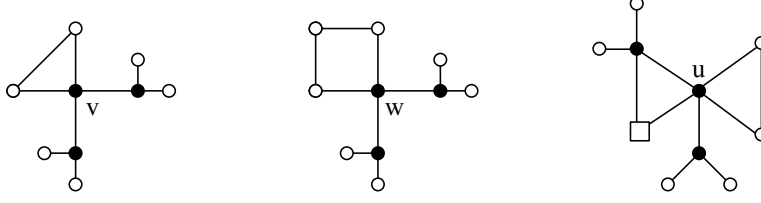


Figure 1: A light 4-vertex v , a soft 4-vertex w and a bad 5-vertex u .

3 Proof of Theorem 1

The proof of Theorem 1 is done by reducible configurations and discharging procedure. Suppose the theorem is not true. Let G be a counterexample with the least number of vertices and edges embedded in the plane. Thus, G is connected. We will apply a discharging procedure to reach a contradiction.

We first define a weight function ω on the vertices and faces of G by letting $\omega(v) = 3d(v) - 10$ if $v \in V(G)$ and $\omega(f) = 2d(f) - 10$ if $f \in F(G)$. It follows from Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces is equal to

$$\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -20.$$

We then design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new weight function satisfies $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction,

$$-20 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \geq 0$$

and hence demonstrates that no such counterexample can exist.

3.1 Reducible configurations of G

In this section, we will establish structural properties of G . More precisely, we prove that some configurations are reducible. Namely, they cannot appear in G because of the minimality of G . Since G does not contain a 4-cycle adjacent to an i -cycle, where $i = 3, 4$, by hypothesis, the following fact is easy to observe and will be frequently used throughout this paper without further notice.

Observation 1 G does not contain the following structures:

- (a) *adjacent 3-cycles;*
- (b) *a 4-cycle adjacent to a 3-cycle;*
- (c) *a 4-cycle adjacent to a 4-cycle.*

We first present Lemma 1, whose proof was provided in [6].

Lemma 1 [6]

- (A1) $\delta(G) \geq 3$.
- (A2) *No two adjacent 3-vertices.*
- (A3) *There is no $(3, 4, 4)$ -face.*

Before showing Lemmas 2-7, we need to introduce some useful concepts, which were firstly defined by Zhang in [12].

Definition 1 For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S . We simply write $G - S = G[V(G) \setminus S]$. Let L be an arbitrary list assignment of G , and π be an $(L, 1)^*$ -coloring of $G - S$. For each $v \in S$, let $L_\pi(v) = L(v) \setminus \{\pi(u) : u \in N_{G-S}(v)\}$, and we call L_π an *induced assignment* of $G[S]$ from π . We also say that π can be extended to G if $G[S]$ admits an $(L_\pi, 1)^*$ -coloring.

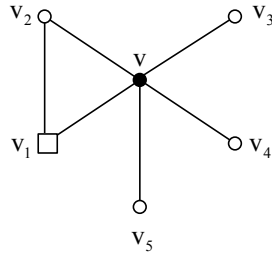


Figure 2: The configuration (Q) in Lemma 2.

Lemma 2 Suppose that G contains the configuration (Q), depicted in Figure 2. Let π be an $(L, 1)^*$ -coloring of $G - S$, where $S = \{v, v_1, v_2, v_3, v_4\}$. Denote by L_π an induced list assignment of $G[S]$. If $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 4\}$, then π can be extended to the whole graph G .

Proof. Since $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 4\}$, we can color each v_i with a color $\pi(v_i) \in L_\pi(v_i)$ properly. Note that $|L_\pi(v)| \geq 2$. If there exists a color in $L_\pi(v)$ which appears at most once on the set $\{v_1, v_2, v_3, v_4\}$, then we assign such a color to v . It is easy to check that the resulting coloring is

an $(L, 1)^*$ -coloring and thus we are done. Otherwise, w.l.o.g., suppose $L(v) = \{1, 2, 3\}$, $\pi(v_5) = 1$, and each color in $\{2, 3\}$ appears exactly twice on the set $\{v_1, v_2, v_3, v_4\}$. W.l.o.g., suppose $\pi(v_1) = 2$.

By definition, we see that v_1 is either a 3-vertex or a light 4-vertex. We label two steps in the proof for future reference.

(i) If $d(v_1) = 3$, then $|L_\pi(v_1)| \geq 2$. We may assign color 2 to v and then recolor v_1 with a color in $L_\pi(v_1) \setminus \{2\}$.

(ii) If v_1 is a light 4-vertex, denote by x_1, y_1 the other two neighbors which are different from v and v_2 . Erase the color of v_1 , color v with 2, and recolor x_1 and y_1 with a color different from its neighbors. We can do this since $d(x_1) = d(y_1) = 3$ by definition. Next, we will show how to extend the resulting coloring, denoted by π' , to G . If $\pi'(v_2) \notin \{\pi'(x_1), \pi'(y_1)\}$, then color v_1 with a color in $L(v_1) \setminus \{2, \pi'(x_1)\}$. Otherwise, we color v_1 with a color in $L(v_1) \setminus \{2, \pi'(v_2)\}$. In each case, one can easily check that the obtained coloring of G is an $(L, 1)^*$ -coloring.

Therefore, we complete the proof of Lemma 2. \square

Lemma 3 *G satisfies the following.*

(B1) *A 4-vertex is adjacent to at most two 3-vertices.*

(B2) *There is no $(4^-, 4^-, 4^-)$ -face.*

(B3) *There is no $(5^+, 4, 4)$ -face which is incident to two light 4-vertices.*

(B4) *There is no 5-vertex incident to a $(5, *, 4)$ -face f and adjacent to two 3-vertices not on $b(f)$.*

(B5) *There is no 6-vertex incident to two $(6, 4^-, 4^-)$ -faces and one $(6, *, 4)$ -face.*

Proof. Let L be a list assignment such that $|L(v)| = 3$ for all $v \in V(G)$. We make use of contradiction to show (B1)-(B5).

(B1) Suppose that v is adjacent to three 3-vertices v_1, v_2 and v_3 . Denote $G' = G - \{v, v_1, v_2, v_3\}$. By the minimality of G , G' admits an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. It is easy to deduce that $|L_\pi(v)| \geq 2$ and $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, 2, 3\}$. So for each v_i , we assign the color $\pi(v_i) \in L_\pi(v_i)$ to it. Now we observe that there exists a color in $L_\pi(v)$ appearing at most once on the set $\{v_1, v_2, v_3\}$. We color v with such a color. The obtained coloring is an $(L, 1)^*$ -coloring of G . This contradicts the choice of G .

(B2) It suffices to prove that G does not contain a $(4, 4, 4)$ -face by (A3). Suppose $f = [v_1 v_2 v_3]$ is a 3-face with $d(v_1) = d(v_2) = d(v_3) = 4$. For each $i \in \{1, 2, 3\}$, let x_i, y_i denote the other two neighbors of v_i not on $b(f)$. Denote by G' the graph obtained from G by deleting

edge v_1v_2 . By the minimality of G , G' has an $(L, 1)^*$ -coloring π . If $\pi(v_1) \neq \pi(v_2)$, then G itself is $(L, 1)^*$ -colorable and thus we are done. Otherwise, suppose $\pi(v_1) = \pi(v_2)$. If π is not an $(L, 1)^*$ -coloring of the whole graph G , then without loss of generality, assume that $\pi(v_1) = \pi(v_2) = \pi(x_1) = 1$ and $\pi(v_3) = 2$. Moreover, none of x_1 's neighbors except v_1 is colored with 1. First, we recolor each v_i with a color $\pi'(v_i)$ in $L(v_i) \setminus \{\pi(x_i), \pi(y_i)\}$, where $i \in \{1, 2, 3\}$. We should point out that $\pi'(v_i)$ may be the same as $\pi(v_i)$, but it does not matter. Note that if at most two of $\pi'(v_1), \pi'(v_2), \pi'(v_3)$ are equal then the resulting coloring is an $(L, 1)^*$ -coloring and thus we are done. Otherwise, suppose that $\pi'(v_1) = \pi'(v_2) = \pi'(v_3)$. Since $\pi'(v_1) \neq 1$ and $1 \in L(v_1)$, we may further reassign color 1 to v_1 to obtain an $(L, 1)^*$ -coloring of G . This contradicts the choice of G .

(B3) Suppose $f = [v_1v_2v_3]$ is a $(5^+, 4, 4)$ -face incident to two light 4-vertices v_2 and v_3 . By definition, we see that each v_i ($i \in \{2, 3\}$) is incident to two other 3-vertices, denoted by x_i and y_i , which are not on $b(f)$. Let G' denote the graph obtained from G by deleting edge v_2v_3 . Obviously, G' has an $(L, 1)^*$ -coloring π by the minimality of G . Similarly, if $\pi(v_2) \neq \pi(v_3)$, then G itself is $(L, 1)^*$ -colorable and thus we are done. Otherwise, suppose $\pi(v_2) = \pi(v_3)$. If π is not an $(L, 1)^*$ -coloring of G , then w.l.o.g., assume that $\pi(v_2) = \pi(v_3) = \pi(x_2) = 1$ and $\pi(v_1) = 2$. Erase the color of v_2 and recolor y_2 with a color $a \in L(y_2)$ different from its neighbors. If $L(v_2) \neq \{1, 2, a\}$, then color v_2 with a color in $L(v_2) \setminus \{1, 2, a\}$. Otherwise, color v_2 with a . It is easy to verify that the resulting coloring is an $(L, 1)^*$ -coloring of G , which is a contradiction.

(B4) Suppose that a 5-vertex v is incident to a $(5, *, 4)$ -face $f_1 = [vv_1v_2]$ and adjacent to two 3-vertices v_3 and v_4 . Let $G' = G - \{v, v_1, v_2, v_3, v_4\}$. By the minimality of G , G' has an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. Obviously, $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 4\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, π can be extended to G , which is a contradiction.

(B5) Suppose that a 6-vertex v is incident to two $(6, 4^-, 4^-)$ -faces f_1, f_3 and one $(6, *, 4)$ -face f_5 such that $d(v_i) \leq 4$ for each $i = \{1, 2, 3, 4\}$, $d(v_6) = 4$ and v_5 is an \mathcal{S} -vertex. Namely, v_5 is either a 3-vertex or a light 4-vertex. Let $G' = G - \{v, v_1, v_2, \dots, v_6\}$. By minimality, G' admits an $(L, 1)^*$ -coloring π . Denote by L_π an induced list assignment of $G - G'$. It is easy to verify that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 6\}$ and $|L_\pi(v)| \geq 3$. So we can color v_i with $\pi(v_i) \in L_\pi(v_i)$ for each $i \in \{1, 2, \dots, 6\}$. If there exists a color $a \in L_\pi(v)$ appearing at most once on the set $\{v_1, v_2, \dots, v_6\}$, then we further assign color a to v and thus obtain an $(L, 1)^*$ -coloring of G .

Otherwise, each color in $L_\pi(v)$ appears exactly twice on the set $\{v_1, v_2, \dots, v_6\}$. Since v_5 is an \mathcal{S} -vertex, we can apply versions of arguments (i) and (ii) in the proof of Lemma 2 to obtain an $(L, 1)^*$ -coloring of G . \square

Lemma 4 Suppose that $f = [uvxy]$ is a $(3, 4, m, 4)$ -face. Then

(F1) $m \neq 3$.

(F2) x cannot be a soft 4-vertex.

Proof. (F1) Suppose to the contrary that $m = 3$. Let $G' = G - \{u, v, x, y\}$. By the minimality of G , G' admits an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. Notice that $|L_\pi(y)| \geq 1$, $|L_\pi(v)| \geq 1$, $|L_\pi(u)| \geq 2$ and $|L_\pi(x)| \geq 2$. First, we color v with $a \in L_\pi(v)$ and color y with $b \in L_\pi(y)$. Then color u with $c \in L_\pi(u) \setminus \{a\}$ and x with $d \in L_\pi(x) \setminus \{b\}$. One can easily check that the resulting coloring of G is an $(L, 1)^*$ -coloring. This contradicts the assumption of G .

(F2) Suppose to the contrary that x is a soft 4-vertex. By definition, x has other two neighbors whose degree are both 3, say x_1 and x_2 . Observe that neither x_1 nor x_2 is on $b(f)$. Let $G' = G - \{u, v, x, y, x_1, x_2\}$. Obviously, G' admits an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. For each $w \in \{v, y, x_1, x_2\}$, we deduce that $|L_\pi(w)| \geq 1$. Moreover, $|L_\pi(u)| \geq 2$. We first color w with $\pi(w) \in L_\pi(w)$ and color u with a color in $L_\pi(u) \setminus \{\pi(v)\}$. If at least one of x_1 and x_2 has the same color as $\pi(v)$, we can color x with a color different from that of v and y . Otherwise, we can color x with a color different from x_1 and y . Therefore, we achieve an $(L, 1)^*$ -coloring of G , which is a contradiction. \square

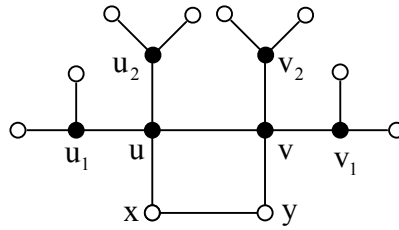


Figure 3: Adjacent soft 4-vertices u and v .

Lemma 5 There is no adjacent soft 4-vertices.

Proof. Suppose to the contrary that u and v are adjacent soft 4-vertices such that $[uxyv]$ is a 4-face and u_1, u_2, v_1, v_2 are 3-vertices, which is depicted in Figure 3. By Observation 1(b), u_i cannot be coincided with v_j , where $i, j \in \{1, 2\}$. Let $G' = G - \{u_1, u_2, v_1, v_2, u, v\}$. For each $i \in \{1, 2\}$,

we color u_i and v_i with a color in $L_\pi(u_i)$ and $L_\pi(v_i)$, respectively. If $L(u) \neq \{\pi(x), \pi(u_1), \pi(u_2)\}$, then color u with $a \in L(u) \setminus \{\pi(x), \pi(u_1), \pi(u_2)\}$. It is easy to see that there exists at least one color in $L(v) \setminus \{\pi(y)\}$ which appears at most once on the set $\{u, v_1, v_2\}$. So we may assign such a color to v . Now suppose that $L(u) = \{\pi(x), \pi(u_1), \pi(u_2)\}$. By symmetry, we may suppose that $L(v) = \{\pi(y), \pi(v_1), \pi(v_2)\}$. This implies that $\pi(v_1) \neq \pi(v_2)$. Thus, we can first color u with $\pi(u_1)$ and then assign a color in $L(v) \setminus \{\pi(u_1), \pi(y)\}$ to v . \square

Lemma 6 *Suppose v is a 5-vertex incident to two 3-faces $f_1 = [vv_1v_2]$ and $f_3 = [vv_3v_4]$. Let v_5 be the neighbour of v not belonging to f_1 and f_3 . Then the following holds.*

- (C1) *If f_1 and f_3 are both $(5, 4^-, 4^-)$ -faces, then $d(v_5) \geq 4$.*
- (C2) *If f_1 is a $(5, *, 4)$ -face and f_3 is a $(5, *, 4^+)$ -face, then $d(v_5) \geq 4$.*
- (C3) *f_1 and f_3 cannot be both $(5, *, 4)$ -faces.*

Proof. In each of following cases, we will show that an $(L, 1)^*$ -coloring of $G' \subset G$ can be extended to G , which is a contradiction.

- (C1) We only need to show that $d(v_5) \neq 3$ since $\delta(G) \geq 3$ by (A1). Suppose that v_5 is a 3-vertex. Let $G' = G - \{v, v_1, \dots, v_5\}$. By the minimality of G , G' has an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. It is easy to deduce that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 5\}$ and $|L_\pi(v)| \geq 3$. So we first color each v_i with $\pi(v_i) \in L_\pi(v_i)$. Observe that there exists a color $a \in L_\pi(v)$ that appears at most once on the set $\{v_1, v_2, \dots, v_5\}$. Therefore, we can color v with a to obtain an $(L, 1)^*$ -coloring of G .
- (C2) Suppose that $d(v_2) = 4$, $d(v_5) = 3$ and v_1 and v_3 are both \mathcal{S} -vertices. By definition, we see that v_i is either a 3-vertex or a light 4-vertex, where $i \in \{1, 3\}$. Let $G' = G - \{v, v_1, v_2, v_3, v_5\}$. By the minimality of G , G' has an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. The proof is split into two cases in light of the conditions of v_3 .
 - Assume v_3 is a 3-vertex. It is easy to calculate that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, 2, 3, 5\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, π can be extended to G .
 - Assume v_3 is a light 4-vertex. By definition, let x_3, y_3 denote the other two neighbors of v_3 not on $b(f_3)$. Recolor x_3 and y_3 with a color different from its neighbors. Next, we will show how to extend the resulting coloring π' to G . Denote $L_{\pi'}$ be the induced assignment of $G - G'$. Notice that $|L_{\pi'}(v_i)| \geq 1$ for each $i \in \{1, 2, 5\}$. If $|L_{\pi'}(v_3)| \geq 1$, then by Lemma 2, π' can be extended to G . Otherwise, we derive that $L(v_3) =$

$\{\pi'(x_3), \pi'(y_3), \pi'(v_4)\}$. First we assign a color in $L_{\pi'}(v_i)$ to each v_i , where $i \in \{1, 2, 5\}$. It is easy to see that there is at least one color, say a , belonging to $L(v) \setminus \{\pi'(v_4)\}$ that appears at most once on the set $\{v_1, v_2, v_5\}$. We assign such a color a to v . Then color v_3 with a color in $\{\pi'(x_3), \pi'(y_3)\}$ but different from a .

(C3) Suppose that f_1 and f_3 are both $(5, *, 4)$ -faces such that $d(v_2) = d(v_4) = 4$ and v_1 and v_3 are \mathcal{S} -vertices. Let $G' = G - \{v, v_1, \dots, v_4\}$. Obviously, G' has an $(L, 1)^*$ -coloring π by the minimality of G . Let L_π be an induced list assignment of $G - G'$. We assert that v_i satisfies that $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, \dots, 4\}$ and $|L_\pi(v)| \geq 2$. By Lemma 2, we can extend π to the whole graph G successfully. \square

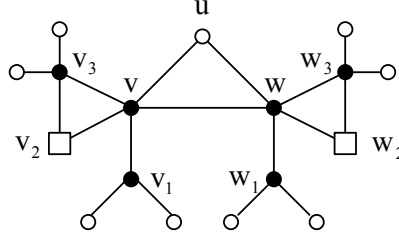


Figure 4: The configuration in Lemma 7.

Lemma 7 *There is no 3-face incident to two bad 5-vertices.*

Proof. Suppose to the contrary that there is a 3-face $[uvw]$ incident to two bad 5-vertices v and w , depicted in Figure 4. Let $G' = G - \{v, w, v_1, v_2, v_3, w_1, w_2, w_3\}$. By the minimality of G , G' has an $(L, 1)^*$ -coloring π . Let L_π be an induced list assignment of $G - G'$. Since each w_i has at most two neighbors in G' , we deduce that $|L_\pi(w_i)| \geq 1$ for each $i \in \{1, 2, 3\}$. So we first color each w_i with a color $\pi(w_i) \in L_\pi(w_i)$. If $|L_\pi(w)| \geq 1$, namely $L(w) \neq \{\pi(u), \pi(w_1), \pi(w_2), \pi(w_3)\}$, then by Lemma 2 we may easily extend π to G , since $|L_\pi(v_i)| \geq 1$ for each $i \in \{1, 2, 3\}$. Otherwise, we deduce that there exists a color a in $L(w) \setminus \{\pi(u)\}$ that is the same as $\pi(w_{i^*})$ for some fixed $i^* \in \{1, 2, 3\}$. Color w with a and v_i with a color $\pi(v_i) \in L_\pi(v_i)$ firstly, where $i \in \{1, 2, 3\}$. For our simplicity, denote $V^* = \{v_1, v_2, v_3, w\}$.

First, suppose that there is a color, say $b \in L(v) \setminus \{\pi(u)\}$, appearing at most once on the set V^* . We assign such a color b to v . If $b \neq a$, the obtained coloring is obviously an $(L, 1)^*$ -coloring. Otherwise, assume that $b = a$. Now we erase the color a from w . One may check that the resulting coloring, say π' , satisfies that each of v, w_1, w_2, w_3 has at least one possible color in $G - G'$. In other words, $|L_{\pi'}(s)| \geq 1$ for each $s \in \{v, w_1, w_2, w_3\}$. Hence, by Lemma 2, we can easily extend π' to G .

Now, w.l.o.g., suppose that $L(v) = \{1, 2, 3\}$, $\pi(u) = 1$, $\pi(w) = 2$ and each color in $\{2, 3\}$ appears exactly twice on the set V^* . It implies that $\pi(v_1) \in \{2, 3\}$. We apply versions of discussion (i) and (ii) in the proof of Lemma 2. After doing that, one may check that now v is colored with $\pi(v_2)$ and v_1 is recolored with a new color, say α . There are two cases left to discuss: if $\pi(v_2) = 3$, namely the new color of v is 3, then the obtained coloring is an $(L, 1)^*$ -coloring and thus we are done; otherwise, we uncolor w . Again, it is easy to see that the resulting coloring, say π'' , satisfies that $|L_{\pi''}(s)| \geq 1$ for each $s \in \{v, w_1, w_2, w_3\}$. Therefore, we can easily extend π'' to G successfully by Lemma 2. \square

3.2 Discharging progress

We now apply a discharging procedure to reach a contradiction. Suppose that u is adjacent to a 3-vertex v such that uv is not incident to any 3-faces. We call v a *free* 3-vertex if $t(v) = 0$ and a *pendant* 3-vertex if $t(v) = 1$. For simplicity, we use $\nu_3(u)$ to denote the number of free 3-vertices adjacent to u and $p_3(u)$ to denote the number of pendant 3-vertices of u . Suppose that v is a soft 4-vertex such that $f_1 = [vv_1uv_2]$ is a 4-face and $d(v_3) = d(v_4) = 3$. If the opposite face to f_1 via v , i.e., f_3 , is of degree at least 5, then we call v a *weak* 4-vertex. We notice that every weak 4-vertex is soft but not vice versa.

For $x \in V(G)$ and $y \in F(G)$, let $\tau(x \rightarrow y)$ denote the amount of weights transferred from x to y . Suppose that $f = [v_1v_2v_3]$ is a 3-face. We use $(d(v_1), d(v_2), d(v_3)) \rightarrow (c_1, c_2, c_3)$ to denote $\tau(v_i \rightarrow f) = c_i$ for $i = 1, 2, 3$. Our discharging rules are defined as follows:

(R1) Let $f = [v_1v_2v_3]$ be a 3-face. We set

$$(R1.1) \quad (3, 4, 5^+) \rightarrow (0, 1, 3);$$

$$(R1.2) \quad (3, 5^+, 5^+) \rightarrow (0, 2, 2);$$

(R1.3)

$$(4, 4, 5^+) \rightarrow \begin{cases} (0, 1, 3) & \text{if } v_1 \text{ is a light 4-vertex;} \\ (1, 1, 2) & \text{if neither } v_1 \text{ nor } v_2 \text{ is a light 4-vertex.} \end{cases}$$

(R1.4)

$$(4, 5^+, 5^+) \rightarrow \begin{cases} (1, 1, 2) & \text{if } v_2 \text{ is a bad 5-vertex;} \\ (0, 2, 2) & \text{if neither } v_2 \text{ nor } v_3 \text{ is a bad 5-vertex.} \end{cases}$$

(R1.5)

$$(5^+, 5^+, 5^+) \rightarrow \begin{cases} (1, \frac{3}{2}, \frac{3}{2}) & \text{if } v_1 \text{ is a bad 5-vertex;} \\ (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) & \text{if none of } v_1, v_2, v_3 \text{ is a bad 5-vertex.} \end{cases}$$

(R2) Suppose that v is a 5^+ -vertex incident to a 4-face $f = [vv_1uv_2]$. Then

(R2.1) $\tau(v \rightarrow f) = 1$ if $d(v_1) \geq 4$ and $d(v_2) \geq 4$;

(R2.2) $\tau(v \rightarrow f) = \frac{4}{3}$ otherwise.

(R3) Suppose that v is a non-weak 4-vertex incident to a 4-face $f = [vv_1uv_2]$.

(R3.1) Assume $d(v_1) = d(v_2) = 3$. Then

(R3.1.1) $\tau(v \rightarrow f) = \frac{4}{3}$ if the opposite face to f via v is of degree 3;

(R3.1.2) $\tau(v \rightarrow f) = \frac{2}{3}$ otherwise.

(R3.2) Assume $d(v_1) \geq 4$ and $d(v_2) \geq 4$. Then

(R3.2.1) $\tau(v \rightarrow f) = 1$ if at least one of v_1 and v_2 is a soft 4-vertex;

(R3.2.2) $\tau(v \rightarrow f) = \frac{2}{3}$ otherwise.

(R3.3) Assume $d(v_1) = 3$ and $d(v_2) \geq 4$. Then $\tau(v \rightarrow f) = \frac{2}{3}$.

(R4) Every 4^+ -vertex sends 1 to each pendant 3-vertex and $\frac{1}{3}$ to each free 3-vertex.

According to (R3), we notice that a weak 4-vertex does not send any charge.

We first consider the faces. Let f be a k -face.

Case $k = 3$. Initially $\omega(f) = -4$. Let $f = [v_1v_2v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. By (A1), $d(v_1) \geq 3$. If $d(v_1) = 3$, then $d(v_2) \geq 4$ by (A2). Together with (B2), we deduce that f is either a $(3, 4, 5^+)$ -face, a $(3, 5^+, 5^+)$ -face, a $(4, 4, 5^+)$ -face, a $(4, 5^+, 5^+)$ -face or a $(5^+, 5^+, 5^+)$ -face. It follows from (B3) and Lemma 7 that every possibility is indeed covered by rule (R1). Obviously, f takes charge 4 in total from its incident vertices. Therefore, $\omega^*(f) = -4 + 4 = 0$.

Case $k = 4$. Clearly, $w(f) = -2$. Assume that $f = [vxuy]$ is a 4-face. By (A2), there are no adjacent 3-vertices in G . It follows that f is incident to at most two 3-vertices. By symmetry, we have to discuss three cases depending on the conditions of these 3-vertices.

- $d(x) = d(y) = 3$. By (F1), we deduce that at least one of u and v is of degree at least 5. Moreover, if one of u and v is a 4-vertex, say v , we claim that v cannot be weak by definition and (B1). Hence, $\omega^*(f) \geq -2 + \frac{4}{3} + \frac{2}{3} = 0$ by (R2) and (R3).
- $d(x) = 3$ and $d(y) \geq 4$. Note that u and v are both 4^+ -vertices. Similarly, neither u nor v can be a weak 4-vertex. It follows from (R3.3) and (R2) that each of u and v sends charge at least $\frac{2}{3}$ to f . So if one of them is a 5^+ -vertex, say v , then by (R2) we have that $\tau(v \rightarrow f) = \frac{4}{3}$ and thus f gets $\frac{2}{3} + \frac{4}{3} = 2$ in total from incident vertices of f . Otherwise, suppose $d(u) = d(v) = 4$. Now by (F2), y cannot be a soft 4-vertex and thus not weak. Hence, $\omega^*(f) \geq -2 + \frac{2}{3} \times 3 = 0$ by (R3.2).

- $d(x) \geq 4$ and $d(y) \geq 4$. Namely, f is a $(4^+, 4^+, 4^+, 4^+)$ -face. If at most one of u, v, x, y is a weak 4-vertex, then $\omega^*(f) \geq -2 + \frac{2}{3} \times 3 = 0$. Otherwise, by Lemma 5, assume that v and u are weak 4-vertices and thus soft. We see that $\tau(x \rightarrow f) = \tau(y \rightarrow f) = 1$ by (R3.2.1) and (R2.1) which implies that $\omega^*(f) \geq -2 + 1 \times 2 = 0$.

Case $k \geq 5$. Then $\omega^*(f) = \omega(f) = 2d(f) - 10 \geq 0$.

Now we consider the vertices. Let v be a k -vertex with $k \geq 3$ by (A1). For $v \in V(G)$, we use $m_4(v)$ to denote the number of 4-faces incident to v . So by Observation 1 (a) and (b), we derive that $t(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ and $m_4(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. Furthermore, $t(v) + m_4(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ by Observation 1 (c).

Observation 2 Suppose v is a 4^+ -vertex which is incident to a 3-face f . Then, by (R1), we have the following:

- (a) $\tau(v \rightarrow f) \leq 1$ if $d(v) = 4$;
- (b) $\tau(v \rightarrow f) \in \{3, 2, \frac{3}{2}, \frac{4}{3}, 1\}$ if $d(v) \geq 5$; moreover, if $\tau(v \rightarrow f) = 3$ then f is a $(5^+, *, 4)$ -face.

Case $k = 3$. Then $\omega(v) = -1$. Clearly, $t(v) \leq 1$. If $t(v) = 1$, then there exists a neighbor of v , say u , so that v is a pendant 3-vertex of u . By (A2), $d(u) \geq 4$. Thus, $\omega^*(v) = -1 + 1 = 0$ by (R4). Otherwise, we obtain that $\omega^*(v) = -1 + \frac{1}{3} \times 3 = 0$ by (R4).

Case $k = 4$. Then $\omega(v) = 2$. Note that $t(v) \leq 2$. If $t(v) = 2$, then $m_4(v) = 0$ and $p_3(v) = 0$. So $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by Observation 2 (a). If $t(v) = 0$, then $n_3(v) \leq 2$ by (B1) and $m_4(v) \leq 2$. We need to consider following cases.

- $m_4(v) = 2$. W.l.o.g., assume that $f_1 = [vv_1uv_2]$ and $f_3 = [vv_3wv_4]$ are incident 4-faces. Obviously, $p_3(v) = 0$ by Observation 1 (b). However, $\nu_3(v) \leq 2$ by (B1). By (R3), v sends charge at most 1 to f_i , where $i = 1, 3$. If $n_3(v) = 0$, then $\nu_3(v) = 0$ and thus $\omega^*(v) \geq 2 - 1 \times 2 = 0$. If $n_3(v) = 1$, say v_1 is a 3-vertex, then $\tau(v \rightarrow f_1) \leq \frac{2}{3}$ by (R3.3) and thus $\omega^*(v) \geq 2 - \frac{2}{3} - 1 - \frac{1}{3} = 0$ by (R4). Now suppose that $n_3(v) = 2$. By symmetry, we have two cases depending on the conditions of these two 3-vertices. If $d(v_1) = d(v_2) = 3$, then $\tau(v \rightarrow f_1) = \frac{2}{3}$ by (R3.1.2). By (B1), v_3 and v_4 are both 4^+ -vertices. Moreover, neither v_3 nor v_4 is a soft 4-vertex according to Lemma 5. So by (R3.2.2), $\tau(v \rightarrow f_3) \leq \frac{2}{3}$. Hence $\omega^*(v) \geq 2 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \times 2 = 0$. Otherwise, suppose that $d(v_i) = d(v_j) = 3$, where $i \in \{1, 2\}$ and $j \in \{3, 4\}$. We derive that $\omega^*(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R3.3).
- $m_4(v) = 1$. W.l.o.g., assume that $d(f_1) = 4$. This implies that $d(f_3) \geq 5$. Again, $\tau(v \rightarrow f_1) \leq 1$ by (R3). If $n_3(v) \leq 1$ then we have that $\omega^*(v) \geq 2 - 1 - 1 = 0$ by (R4). So in what follows, we

assume that $n_3(v) = 2$. If $d(v_3) = d(v_4) = 3$ then v is a weak 4-vertex, implying that v sends nothing to f_1 . So $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R4). If $d(v_1) = d(v_2) = 3$, then $p_3(v) = 0$ by Observation 1 (b). We deduce that $\omega^*(v) \geq 2 - \frac{2}{3} - \frac{1}{3} \times 2 = \frac{2}{3}$ by (R3.1.2) and (R4). Otherwise, suppose $d(v_i) = d(v_j) = 3$, where $i \in \{1, 2\}$ and $j \in \{3, 4\}$. It follows immediately from (R3.3) and (R4) that $\omega^*(v) \geq 2 - \frac{2}{3} - 1 - \frac{1}{3} = 0$.

- $m_4(v) = 0$. Obviously, $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R4).

Now, in the following, we consider the case $t(v) = 1$. Assume that f_1 is a 3-face. By (A1) and (B2), f_1 is either a $(4, 3, 5^+)$ -face, a $(4, 4, 5^+)$ -face or a $(4, 5^+, 5^+)$ -face. Observe that $m_4(v) \leq 1$. First assume that $m_4(v) = 0$. If f_1 is a $(4, 3, 5^+)$ -face, then $p_3(v) \leq 1$ by (B1) and hence $\omega^*(v) \geq 2 - 1 - 1 = 0$ by Observation 2 (a) and (R2). Next suppose that f_1 is a $(4, 4, 5^+)$ -face. If $n_3(v) = 2$, then v is a light 4-vertex. By (R1.3), we see that v sends nothing to f_1 and therefore $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R4). Otherwise, at most one of v_3, v_4 is a 3-vertex and hence $\omega^*(v) \geq 2 - 1 - 1 = 0$ by Observation 2 (a) and (R4). Finally, we suppose that f_1 is a $(4, 5^+, 5^+)$ -face. If neither v_1 nor v_2 is a bad 5-vertex, then v sends nothing to f_1 by (R1.4) and thus $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R4). Otherwise, one of v_1 and v_2 is a bad 5-vertex. It follows directly from (C2) that $n_3(v) \leq 1$. Therefore, $\omega^*(v) \geq 2 - 1 - 1 = 0$ by (R2). Now suppose that $m_4(v) = 1$. By Observation 1 (c), we may assume that $f_3 = [vv_3wv_4]$ is a 4-face. In this case, $p_3(v) = 0$. If $d(v_3) = d(v_4) = 3$, then $\tau(v \rightarrow f_3) = \frac{4}{3}$ by (R3.1.1). It follows from (B1) and (C2) that f is neither a $(4, 3, 5^+)$ -face nor a $(4, 5, 5^+)$ -face such that v_2 is a bad 5-vertex. So we deduce that f_1 gets nothing from v by (R1.3), which implies that $\omega^*(v) \geq 2 - \frac{4}{3} - \frac{1}{3} \times 2 = 0$. If exactly one of v_3, v_4 is a 3-vertex, then $\tau(v \rightarrow f_3) \leq \frac{2}{3}$ by (R3.3). Thus, $\omega^*(v) \geq 2 - 1 - \frac{2}{3} - \frac{1}{3} = 0$ by Observation 2 (a) and (R4). Otherwise, we suppose that v_3, v_4 are both of degree at least 4. In this case, $\nu_3(v) = 0$ and hence $\omega^*(v) \geq 2 - 1 - 1 = 0$ by (R3.2) and Observation 2 (a).

Case $k = 5$. Then $\omega(v) = 5$. Also, $t(v) \leq 2$. we have three cases to discuss.

Assume $t(v) = 0$. If $m_4(v) = 0$, then $\omega^*(v) \geq 5 - 1 \times 5 = 0$ by (R4). If $m_4(v) = 1$, then $p_3(v) \leq 3$. Thus $\omega^*(v) \geq 5 - \frac{4}{3} - 1 \times 3 - 2 \times \frac{1}{3} = 0$ by (R2) and (R4). Now suppose that $m_4(v) = 2$. By Observation 1 (c), we assert that $p_3(v) \leq 1$. So $\omega^*(v) \geq 5 - \frac{4}{3} \times 2 - \frac{1}{3} \times 4 - 1 = 0$.

Next assume $t(v) = 1$, say f_1 . Then $\tau(v \rightarrow f_1) \leq 3$ by Observation 2 (b). Moreover, equality holds iff f_1 is a $(5, *, 4)$ -face. So if $\tau(v \rightarrow f_1) = 3$ then at most one of v_3, v_4, v_5 is a 3-vertex by (B4). Furthermore, $m_4(v) \leq 1$. When $m_4(v) = 0$, we deduce that $\omega^*(v) \geq 5 - 3 - 1 = 1$ by (R4). When $m_4(v) = 1$, by symmetry, say f_3 is a 4-face, we have two cases to discuss: if $p_3(v) = 1$, namely, v_5 is a 3-vertex, then $\tau(v \rightarrow f_3) \leq 1$ by (R2) and neither v_3 nor v_4 takes charge from v . Thus $\omega^*(v) \geq 5 - 3 - 1 - 1 = 0$; otherwise, $p_3(v) = 0$ and we have $\omega^*(v) \geq 5 - 3 - \frac{4}{3} - \frac{1}{3} = \frac{1}{3}$. Now

suppose that $\tau(v \rightarrow f_1) \leq 2$. By (R2) and (R4), $\omega^*(v) \geq 5 - 2 - 1 \times 3 = 0$ if $m_4(v) = 0$ and $\omega^*(v) \geq 5 - 2 - \frac{4}{3} - 1 - 2 \times \frac{1}{3} = 0$ if $m_4(v) = 1$.

Now assume $t(v) = 2$. By symmetry, assume f_1 and f_3 are both 3-faces. Observe that $m_4(v) = 0$. For simplicity, denote $\tau(v \rightarrow f_1) = \sigma_1$ and $\tau(v \rightarrow f_3) = \sigma_2$. Let $\sigma = \max\{\sigma_1, \sigma_2\}$. If $\sigma \leq 2$, then $\omega^*(v) \geq 5 - 2 \times 2 - 1 = 0$ by (R2). Now assume that $\sigma = 3$, i.e., f_1 gets charge 3 from v . It means that f_1 is a $(5, *, 4)$ -face by Observation 2. By (C3), f_3 cannot be a $(5, *, 4)$ -face. This implies that $\sigma_2 \leq 2$. Moreover, if v_5 is a 3-vertex, then f_3 is neither a $(5, *, 4^+)$ -face by (C2) nor a $(5, 4, 4)$ -face by (C1). It follows from (R1.4) and (R1.5) that $\sigma_2 \leq 1$, since v is a bad 5-vertex. Thus, $\omega^*(v) \geq 5 - 3 - 1 - 1 = 0$ by (R2). Otherwise, we easily obtain that $\omega^*(v) \geq 5 - 3 - 2 = 0$.

Case $k \geq 6$. Notice that $t(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. If v is incident to a 4-face f_i , then by (R2) we inspect v sends a charge at most $\frac{4}{3}$ to f_i , while $\frac{1}{3}$ to each of v_i and v_{i+1} . So we may consider v as a vertex which sends charge at most $\frac{4}{3} + 2 \times \frac{1}{3} = 2$ to f_i . So by (R4) and Observation 2, we have

$$\begin{aligned}\omega^*(v) &\geq 3d(v) - 10 - 3t(v) - 2m_4(v) - (d(v) - 2t(v) - 2m_4(v)) \\ &= 2d(v) - 10 - t(v) \equiv \tau(v)\end{aligned}$$

If $d(v) \geq 7$, then $\tau(v) \geq 2d(v) - 10 - \frac{d(v)}{2} = \frac{3}{2}d(v) - 10 \geq \frac{3}{2} \times 7 - 10 = \frac{1}{2} > 0$. Now suppose that $d(v) = 6$. If $t(v) \leq 2$ then $\tau(v) \geq 2 \times 6 - 10 - 2 = 0$. So, in what follows, assume that $t(v) = 3$ and $d(f_i) = 3$ for $i = 1, 3, 5$. Clearly, $m_4(v) = 0$. Similarly, if there are at most two of 3-faces get charge 3×2 in total from v , then $\omega^*(v) \geq 8 - 2 \times 3 - 2 = 0$. Otherwise, suppose $\tau(v \rightarrow f_i) = 3$ for each $i \in \{1, 3, 5\}$. By Observation 2 (b), we assert that f_i is a $(6, *, 4)$ -face. Noting that a $(6, *, 4)$ -face is also a $(6, 4^-, 4^-)$ -face, we may regard v as a 6-vertex which is incident to two $(6, 4^-, 4^-)$ -faces and one $(6, *, 4)$ -face. However, it is impossible by (B5).

Therefore, we complete the proof of Theorem 1. □

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